The Optimum Target Value Under Single and Two-Stage Screenings

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In this paper we consider the problem of determining the optimum target value of the quality characteristic of interest and the screening limits for a correlated variable under single and two-stage screenings. In single-stage screening, inspection is performed directly on the quality characteristic of interest or on a variable that is correlated with the characteristic. In two-stage screening, the correlated variable is inspected first to decide if an item should be accepted, rejected, or whether additional observations should be taken. If additional observations are required, the quality characteristic is then directly observed in order to classify the undecided items. Models are constructed that involve selling and discounted prices as well as production, inspection, and penalty costs for both single and two-stage screenings. Methods for finding the optimum process mean and the screening limits are presented when the quality characteristic and the correlated variable are assumed to be jointly normally distributed. A numerical example is presented.

Introduction

As a result of advances in automated manufacturing systems, sensoring technology, and automatic inspection equipment, full inspections are increasingly being used in industry to improve outgoing product quality. Suppose that there is a lower specification limit L for the quality characteristic Y of interest. All items are subjected to acceptance inspection, and those with $Y < L$ are reprocessed or sold at a discounted price. Such quality characteristics include filling weights and volumes. Items produced by a production process may deviate from the process mean because of variations in materials, labor, and operating conditions. The process mean may be adjusted to a higher value in order to reduce the proportion of the nonconforming items. Using a higher process mean, however, may result in a higher production cost. Therefore, a process parameter μ for the process mean is to be selected so that the expected cost per item is minimized.

Several researchers have studied this problem. Springer (1951) and Bettes (1962) consider a filling process where upper and lower specification limits are given. The optimum target value that minimizes the reprocessing cost and the material costs for overfilled and underfilled items is obtained. Golhar (1987) studies a canning process in which underfilled cans are emptied and refilled so that they can be sold in the primary market. Boucher and Jafari (1991) and Al-Sultan (1994) discuss situations in which the items are subjected to lot-by-lot acceptance sampling rather than complete inspections. Elsayed and Chen (1993) determine optimum levels of process parameters for products with multiple characteristics, and Arcelus and Rahim (1994) develop a model for simul-

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taneously selecting optimum target means for both variable and attribute quality characteristics. Chen and Chung (1996) consider an economic model for determining the most profitable target value and the optimum inspection precision level for a production process.

In all of these studies, inspection is performed directly on the quality characteristic Y of interest (the performance variable). In some situations it is impossible, or not economical, to directly inspect the quality characteristic Y. In such cases, the use of a variable X that is highly correlated with Y is an attractive alternative, especially when inspecting the correlated variable is relatively less expensive than obtaining Y. In a cement plant, for example, a performance measure of interest may be the weight of a cement bag, which is difficult to measure directly due to the high-speed of the packing line. The milampere (mA) of the load cell is strongly correlated with the weight of a cement bag and does not require special effort to measure. Hence, it can be used as the correlated variable (Bai and Lee (1993)). The problem of selecting the cutoff value of X has been studied by many researchers. Bai and Lee (1993) and Tang and Lo (1993) present economic models that determine the process mean and the cutoff value of X when inspection is based on X instead of Y in situations where items with $Y \geq L$ are sold at a fixed price and items with $Y < L$ are scrapped or reprocessed. Hong et al. (1998) consider the problem of jointly determining optimum target values in situations where there are several markets with different price/cost structures. Recently, Linna and Woodall (2001) study the effect of using the correlated variable X in statistical process monitoring of Y.

In applications where quality assurance is critical, the outgoing quality improvement may be more important than the reduction in the inspection cost. Since a correlated variable is not perfectly correlated with Y, some conforming items may be rejected and excluded from shipment while some nonconforming items may be accepted for shipment. These decision errors are likely to occur when the value of a correlated variable is close to the screening limits. Consequently, in this situation there may be an economic advantage to reducing the errors by observing the performance variable even though the inspection may be expensive. Of course, this can only be done when the inspection procedure of the performance variable is not destructive. Based on this, Tang (1988) and Bai et al. (1995) propose economic two-stage screenings where the correlated variable is used in the first stage and the performance variable is used in the second stage.

In this paper, we determine the optimum process mean of the quality characteristic Y of interest and the screening limits of a correlated variable under single and two-stage screenings. In the single-stage screening, inspection is based directly on the performance measure or on a correlated variable. In the two-stage screening, a correlated variable is inspected first to decide whether an item should be accepted, rejected, or whether additional observations should be taken. If additional observations are required, then the performance variable is observed in order to classify the unclassified items. The optimum process mean and screening limits of the correlated variable are jointly determined by maximizing the profit function, which involves selling and discounted prices as well as production, inspection, and penalty costs.

Single-Stage Screening

In this section we present two models, denoted by Models I and II, for the single-stage screening: in Model I, inspection is performed on the performance variable Y; in Model II, inspection is performed on the correlated variable X. Model I is based on Carlsson (1984) with slight modification.

Model I: Inspection is Performed Directly on Y

Let Y be a performance variable representing the quality characteristic of interest, and let L be the lower specification limit of Y. Suppose that Y is normally distributed with an unknown process mean μ_y and known variance σ_y^2. We also assume that there is no measurement error. All items are inspected prior to shipment to determine whether they meet the lower specification limit L on Y. Let a and r denote the selling and discounted prices, respectively, where items with $Y \geq L$ are sold at a fixed price a to the primary market, and items with $Y < L$ are sold at a discounted price $r(< a)$ to the secondary market. The production cost per item is linearly related to Y, that is, $b + cy$ where b and c are constants. Let c_y denote the performance inspection cost per item. The profit function $P(y; \mu_y) \equiv P_{S,I}$ per item is

$$P_{S,I} = \begin{cases} a - b - cy - c_y, & Y \geq L, \\ r - b - cy - c_y, & Y < L. \end{cases} \qquad (1)$$

Similar profit functions are used by several authors. For example, if we let $a = b_a + (c_0 - c_a)L$, $r = b_r + (c_0 - c_a)L$, $b + c_y = b_0$, and $c = c_0 - c_r = c_0 - c_a$, where b_0, b_a, b_r, c_0, c_a and c_r are cost parameters used in Carlsson (1984), the profit function given in Equation (1) is the same as that in Carlsson (1984). The optimum process mean μ_y^* is then

$$\mu_y^* = L - \eta^* \sigma_y, \tag{2}$$

where

$$\eta^* = -(-2\ell n(\sqrt{2\pi} c\sigma_y)/(a-r))^{1/2}$$

for $\sqrt{2\pi} c\sigma_y < a - r$. See Carlsson (1984) for the detailed derivation.

Model II: Inspection is Performed on the Correlated Variable X

Let X be a variable that is positively correlated with Y. If X is negatively correlated with Y, we then use $-X$ as the screening variable rather than X. We assume that, for given $Y = y$, X is normally distributed with mean $\lambda_1 + \lambda_2 y$ and variance σ^2 where λ_1 and λ_2 are known constants. The constant λ_2 is assumed to be positive so that X and Y have a positive correlation. It can be easily shown that (X, Y) follows a bivariate normal distribution with mean vector $(\mu_x = \lambda_1 + \lambda_2\mu_y, \mu_y)$, covariance matrix

$$\begin{pmatrix} \lambda_2\sigma_y^2 + \sigma^2 & \rho\sigma_y\sqrt{\lambda_2^2\sigma_y^2 + \sigma^2} \\ \rho\sigma_y\sqrt{\lambda_2^2\sigma_y^2 + \sigma^2} & \sigma_y^2 \end{pmatrix},$$

and correlation coefficient

$$\rho = \{\lambda_2^2\sigma_y^2/(\lambda_2^2\sigma_y^2 + \sigma^2)\}^{1/2}$$

(see Tang and Lo (1993)). Let $\mathcal{X}$ be the screening limit on the decision variable X. If $X \ge \mathcal{X}$, then we conclude that $Y \ge L$, and the item is sold to the primary market at a fixed price a. Since X is not perfectly correlated with Y, some items with $Y < L$ may be sold to the primary market. An accepted item with $Y < L$ incurs the penalty cost d which includes the cost of identifying and handling the nonconforming item, and the service and replacement cost. If $X < \mathcal{X}$, the item is sold at a discounted price $r(r < a \le d)$. The production cost per item is the same as in the previous model, and c_x denotes the inspection cost per item for X. Then, the profit function per item is

$$P_{S,II} = \begin{cases} a - b - cy - c_x, & X \ge \mathcal{X},\ Y \ge L, \\ a - b - cy - c_x - d, & X \ge \mathcal{X},\ Y < L, \\ r - b - cy - c_x, & X < \mathcal{X}. \end{cases}$$

The expected profit per item is given by

$$\begin{aligned} E(P_{S,II}) = &\int_{\mathcal{X}}^{\infty}\int_{L}^{\infty} (a - b - cy - c_x)\, g(x,y)dydx \\ &+ \int_{\mathcal{X}}^{\infty}\int_{-\infty}^{L} (a - b - cy - c_x - d)\, g(x,y)dydx \\ &+ \int_{-\infty}^{\mathcal{X}}\int_{-\infty}^{\infty} (r - b - cy - c_x)\, g(x,y)dydx, \end{aligned} \tag{3}$$

where $g(x, y)$ is the joint density function of X and Y. Using the relationships

$$\int_{\mathcal{X}}^{\infty}\int_{-\infty}^{L} g(x,y)dydx = \Psi(-\zeta, \eta; -\rho), \tag{4}$$

$$\int_{\mathcal{X}}^{\infty}\int_{L}^{\infty} g(x,y)dydx = \Psi(-\zeta, -\eta; \rho), \tag{5}$$

where

$$\Psi(-\zeta, -\eta; \rho) + \Psi(-\zeta, \eta; -\rho) = \Phi(-\zeta),$$

Equation (3) can be rewritten as

$$\begin{aligned} E(P_{S,II}) = &\, a\Phi(-\zeta) + r\Phi(\zeta) - d\Psi(-\zeta, \eta; -\rho) \\ &- b - c_x - c(L - \eta\sigma_y), \end{aligned}$$

where $\zeta = (\mathcal{X} - \mu_x)/\sigma_x$, $\eta = (L - \mu_y)/\sigma_y$, and $\Phi(\cdot)$ and $\Psi(\cdot)$ are, respectively, the standard normal distribution function and standardized bivariate normal distribution function with correlation coefficient ρ. We will assume that $\eta > 0$; note that $\eta < 0$ if and only if the proportion of defective items $(Y < L)$ is less than 50%. This condition is reasonable in many production processes. Therefore, we are only interested in maximizing the expected profit over the domain $\{(\eta, \zeta) \mid \eta < 0 \text{ and } -\infty < \zeta < \infty\}$.

Using the relationships

$$\frac{\partial \Psi(-\zeta, \eta; \rho)}{\partial \zeta} = -\Phi\left(\frac{\eta + \zeta\rho}{\sqrt{1-\rho^2}}\right)\phi(-\zeta),$$

$$\frac{\partial \Psi(-\zeta, \eta; \rho)}{\partial \eta} = -\Phi\left(\frac{-\zeta - \eta\rho}{\sqrt{1-\rho^2}}\right)\phi(\eta),$$

the first derivatives of $E(P_{S,II})$ with respect to η and ζ are

$$\frac{\partial E(P_{S,II})}{\partial \eta} = -d\Phi\left(\frac{-\zeta + \eta\rho}{\sqrt{1-\rho^2}}\right)\phi(\eta) + c\sigma_y,$$

$$\frac{\partial E(P_{S,II})}{\partial \zeta} = (r-a)\phi(\zeta) + d\Phi\left(\frac{\eta - \zeta\rho}{\sqrt{1-\rho^2}}\right)\phi(\zeta),$$

where $\phi(\cdot)$ is the standard normal density function.

If $E(P_{S,II})$ is a unimodal function of η and ζ, then the optimum values η^* and ζ^* are the values η and ζ satisfying $\partial E(P_{S,II})/\partial\eta = 0$ and $\partial E(P_{S,II})/\partial\zeta = 0$, which give

$$d\Phi\left(\frac{-\zeta + \eta\rho}{\sqrt{1-\rho^2}}\right)\phi(\eta) + c\sigma_y = 0 \qquad (6)$$

$$\zeta^* = \frac{\eta^* - \sqrt{1-\rho^2}\Phi^{-1}((a-r)/d)}{\rho}. \qquad (7)$$

Let the left hand side of Equation (6) be $v_1(\eta)$. Since the derivative of $v_1(\eta)$ is less than zero, then $v_1(\eta)$ is a strictly decreasing function of η for $\eta < 0$. We have $v_1(0) = -d\Phi(-\zeta^*/\sqrt{1-\rho^2})/\sqrt{2\pi} < v_1(\eta) < c\sigma_y = v_1(-\infty)$. This implies that, for $\eta < 0$, Equation (6) has a unique solution. It is difficult to show analytically that $E(P_{S,II})$ is a unimodal function of η and ζ. However, numerical studies over a wide range $[10 \le a/(c\sigma_y) \le 100$, $5 \le r/(c\sigma_y) \le 50$, $25 \le d/(c\sigma_y) \le 250$, $0.6 \le \rho \le 0.99$, $r/(c\sigma_y) < a/(c\sigma_y) \le d/(c\sigma_y)]$ of the parameter values $(r/(c\sigma_y)$, $a/(c\sigma_y)$, $d/(c\sigma_y)$, $\rho)$ indicate that $E(P_{S,II})$ is a unimodal function of η and ζ for $\eta < 0$ and $-\infty < \zeta < \infty$; the domain of interest. The optimum values η^* and ζ^* can be obtained by solving Equations (6) and (7) simultaneously, and a computational approach such as Gauss-Siedel's iterative method can be used to obtain η^* and ζ^*. The optimum process mean μ_y^* is obtained from Equation (2) and the screening limit $\mathcal{X}^*$ of X from $\mathcal{X}^* = \mu_x + \zeta^*\sigma_x$.

Two-Stage Screening (Model III)

Since X is strongly but not perfectly correlated with Y, decision errors of rejecting conforming items or accepting nonconforming items may occur. To reduce the number of these errors, we present a two-stage screening procedure in which a correlated variable X is used in the first stage and a performance variable Y is used in the second stage. The two-stage screening is as follows:

1$^{\text{st}}$stage: Take a measurement x on X for each incoming item. The item is (a) accepted if $x \ge \omega_1$, (b) unclassified if $\omega_2 \le x < \omega_1$, and (c) rejected if $x < \omega_2$, where $\omega_1 \ge \omega_2$.

2$^{\text{nd}}$ stage: Observe y on Y for the unclassified item and (a) accept if $y \ge L$, and (b) reject if $y < L$.

Here, ω_1 and ω_2 are screening limits for X. Note that there are no misclassification errors at the second stage because all the unclassified items are inspected directly using the performance variable. In this model, selling prices, discounted prices and production cost are the same as in Model II. The profit function $P(x, y; \mu_y, \omega_1, \omega_2) \equiv P_{T,III}$ is

$$P_{T,III} = \begin{cases} a - b - cy - c_x, & \mathcal{X} \ge \omega_1,\ Y \ge L \\ a - b - cy - c_x - d, & \mathcal{X} \ge \omega_1,\ Y < L \\ a - b - cy - c_x - c_y, & \omega_2 \le \mathcal{X} < \omega_1,\ Y \ge L \\ r - b - cy - c_x - c_y, & \omega_2 \le \mathcal{X} < \omega_1,\ Y < L \\ r - b - cy - c_x, & \mathcal{X} < \omega_2. \end{cases}$$

The expected profit per item is then given by

$$\begin{aligned} E(P_{T,III}) = {} & a\Phi(-\delta_1) \\ & + a\{\Psi(\delta_1, -\eta; -\rho) - \Psi(\delta_2, -\eta; -\rho)\} \\ & - d\Psi(-\delta_1, \eta; -\rho) + c_y\{\Phi(\delta_2) - \Phi(\delta_1)\} \\ & + r\{\Psi(\delta_1, \eta; \rho) + \Psi(\delta_2, -\eta; -\rho)\} \\ & - (b + c(L - \eta\sigma_y)) - c_x, \end{aligned} \qquad (8)$$

where $\delta_i = (\omega_i - \mu_x)/\sigma_x$, $i = 1, 2$. Note that $\delta_2 < \delta_1$. See Appendix A for the detailed derivation.

The optimum values δ_1^*, δ_2^*, and η^* can be obtained by maximizing $E(P_{T,III})$ over the domain $\{(\eta, \delta_1, \delta_2) \mid \eta < 0,\ -\infty < \delta_1 < \infty$, and $-\infty < \delta_2 < \infty\}$. Again, we restrict $\eta < 0$. For $\eta < 0$, extensive numerical studies suggest that $E(P_{T,III})$ is a unimodal function of η, δ_1, and δ_2. The optimum values η^* and δ_1^* satisfy the conditions $\partial E(P_{T,III})/\partial\eta = 0$ and $\partial E(P_{T,III})/\partial\delta_i = 0$, $i = 1, 2$ as given by Equations (9)-(11):

$$d\tau(-\delta_1^*, \eta^*) + (a-r)\{\tau(\delta_1^*, -\eta^*) - \tau(\delta_2^*, -\eta^*)\} = c\sigma_y, \qquad (9)$$

$$\delta_1^* = \frac{\eta^* - \sqrt{1-\rho^2}\Phi^{-1}(c_y/(d+r-a))}{\rho}, \qquad (10)$$

$$\delta_2^* = \frac{\eta^* + \sqrt{1-\rho^2}\Phi^{-1}(c_y/(a-r))}{\rho}, \qquad (11)$$

where $\tau(\delta, \eta) = \Phi[(\delta + \eta\rho)/\sqrt{1-\rho^2}]\phi(\eta)$. See Appendix B for the detailed derivations. Let the left hand side of Equation (9) be $v_2(\eta^*)$. Since $d \ge a > r$ and $\eta^* < 0$, it is clear that

$$\frac{\partial v_2(\eta^*)}{\partial \eta^*} = (d + r - a)\rho\phi\left(\frac{\delta_1^* - \eta^*\rho}{\sqrt{1-\rho^2}}\right)\left(\frac{\phi(-\eta^*)}{\sqrt{1-\rho^2}}\right)$$

$$+ (a-r)\rho\phi\left(\frac{\delta_2^* - \eta^*\rho}{\sqrt{1-\rho^2}}\right)\left(\frac{\phi(\eta^*)}{\sqrt{1-\rho^2}}\right) - \eta^* c\sigma_y > 0,$$

where $v_2(\eta^*)$ is a strictly increasing function of η^* for $\eta^* < 0$. We then have $v_2(-\infty) = 0 < v_2(\eta^*)$ and

$$v_2(0) = \frac{d\Phi\left(\frac{-\delta_1^*}{\sqrt{1-\rho^2}}\right)}{\sqrt{2\pi}} + \frac{(a-r)\left[\Phi\left(\frac{\delta_1^*}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\delta_2^*}{\sqrt{1-\rho^2}}\right)\right]}{\sqrt{2\pi}}.$$

Therefore, Equations (9)-(11) have a unique solution if

$$\frac{d\Phi\left(\frac{-\delta_1^*}{\sqrt{1-\rho^2}}\right)}{\sqrt{2\pi}} + \frac{(a-r)\left[\Phi\left(\frac{\delta_1^*}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\delta_2^*}{\sqrt{1-\rho^2}}\right)\right]}{\sqrt{2\pi}} \geq c\sigma_y.$$

In many practical production processes, Equation (9) seems to be satisfied because d often far exceeds $c\sigma_y$. Under the assumption that Equation (9) is satisfied, the optimum values η^*, δ_1^* and δ_2^* can be obtained by solving Equations (9)-(11) simultaneously, and, as in Model II, a computational approach such as Gauss-Siedel's method can be used to obtain η^*, δ_1^*, and δ_2^*. The optimum process mean μ_y^* is obtained using Equation (2), and the screening limits ω_1^* and ω_2^* of X are obtained by

$$\omega_i^* = \mu_x + \delta_i\sigma_x, \quad \text{for } i = 1, 2.$$

Numerical Example

In this section we present an example that originally appeared in Bai and Lee (1993) in order to illustrate the optimum solution procedures. Numerical studies are also performed to investigate the effects of σ_y, ρ, and the cost parameters. IMSL (1987) subroutines such as DNORIN, DNORDF, and DBNRDF are used to evaluate the inverse of the standard normal distribution function and standard univariate and bivariate normal distribution functions, respectively.

Consider a cement factory packing plant. The packing operation consists of two processes; a filling process and an inspection process. Each cement bag processed by the filling machine is moved to the loading and dispatching phases on a conveyor belt. Continuous weighing feeders (CWFs) perform inspection. A CWF measures the mA (milampere) X of the load cell of the cement bag that is positively correlated with the weight Y of the cement bag. From theoretical considerations and past experience, it is known that the variance of Y is $\sigma_y^2 = (1.25kg)^2$, and that X for given $Y = y$ is normally distributed with mean $4.0 + 0.08y$ and variance $(0.05mA)^2$. That is, X and Y are jointly normally distributed, with unknown means (μ_x, μ_y), known variances $\sigma_x^2 = (0.112mA)^2$, $\sigma_y^2 = (1.25kg)^2$, and correlation coefficient $\rho = 0.894$. The weight marked on each bag is 40 kg, and it is the lower specification limit. Suppose that the cost components are $a = \$3.0$, $r = \$2.25$, $c_0 = \$0.1$, $c = \$0.06$, $c_y = \$0.04$, $c_x = \$0.004$, and $d = \$6.5$.

For Model I, we obtain $\eta^* = -1.664$ from Equation (2). Hence,

$$\mu_y^* = L - \eta^*\sigma_y = 40.0 - (-1.664 \times 1.25) = 42.079(kg)$$

and

$$E(P_{S,I}) = \$0.299.$$

For Model II, we obtain $\eta^* = -2.305$ and $\zeta^* = -2.003$ from Equations (6) and (7). Therefore the optimum process mean and screening limit for X are

$$\mu_y^* = L - \eta^*\sigma_y = 40.0 - (-2.305 \times 1.25) = 42.882(kg)$$

$$\mathcal{X}^* = \mu_x + \zeta^*\sigma_x = 4.0 + (0.08 \times 42.882) + (-2.003 \times 0.112) = 7.206(mA)$$

and

$$E(P_{S,II}) = \$0.290.$$

For Model III, we obtain $\eta^* = -1.787$, $\delta_1^* = -0.782$, and $\delta_2^* = -2.807$ from Equations (9)-(11). Therefore the optimum process mean and screening limits for X are

$$\mu_y^* = L - \eta^*\sigma_y = 40.0 - (-1.787 \times 1.25) = 42.234(kg)$$

$$\omega_1^* = \mu_x + \delta_1^*\sigma_x = 4.0 + (0.08 \times 42.234) + (-0.782 \times 0.112) = 7.291(mA)$$

$$\omega_2^* = \mu_x + \delta_2^*\sigma_x = 4.0 + (0.08 \times 42.234) + (-2.807 \times 0.112) = 7.064(mA)$$

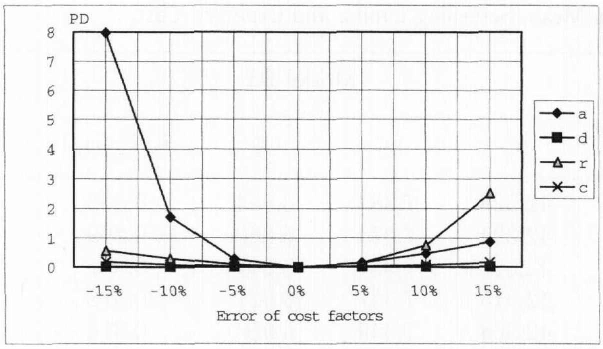

FIGURE 1. Percentage Decrease of Expected Profits for Model III.

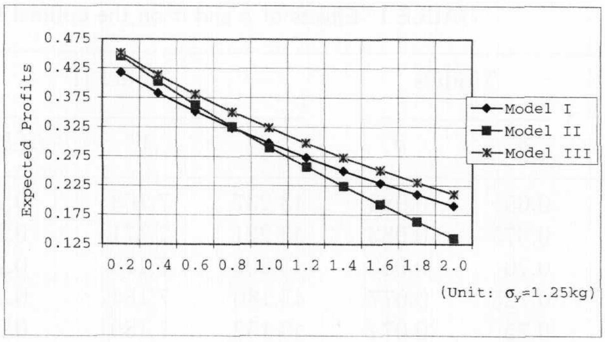

FIGURE 2. Expected Profit as a Function of σ_y.

and

$$E(P_{T,III}) = \$0.3235.$$

These results agree with our intuition that the expected profit for the two-stage screening is higher than the single-stage screenings. However, two-stage screening is somewhat complex to implement. Since a correlated variable is not perfectly correlated with the performance variable, classification errors may occur in Models II and III. Therefore, in applications where quality assurance is critical, Model I is more suitable. In some cases, it is impossible or not economical to directly inspect the performance variable. In these situations, Model II can be effectively used. We also conducted numerical studies to investigate the effects of the parameters (ρ, σ_y, a, c_y, d, and r).

Effects of Using Improper Cost Factors

It is sometimes difficult to obtain accurate estimates of penalty and production costs. If incorrect values for these parameters are used to determine the optimum target values, the calculated expected profit is expected to be smaller than the expected profit with true values. To study the sensitivity of Model III to cost parameters, the percentage decrease (PD) is given in Figure 1 for selected values of a, c, d, and r with remaining parameters fixed as given in our example. The PD is defined as

$$PD = \frac{E(P_{T,III})^* - E(P_{T,III})'}{E(P_{T,III})^*} \times 100(\%),$$

where $E(P_{T,III})^*$ and $E(P_{T,III})'$ are the expected profit obtained by using the true and incorrect cost parameters, respectively. In Figure 1, the negative and positive values on the horizontal axis denote underestimated and overestimated values of cost parameters, respectively. Figure 1 indicates that Model III is very robust to changes in cost parameters except for underestimated values of a and overestimated values of r. It also suggests that similar results can be obtained for Model II, and Model I is robust to changes in cost parameters except for incorrectly estimated values of a and overestimated values of r.

Effect of σ_y

$E(P_{S,I})$, $E(P_{S,II})$, and $E(P_{T,III})$ for the above example are shown in Figures 2 and 3 for selected values of the standard deviation of Y from $0.2\sigma_y$ to $2.0\sigma_y$, where σ_y equals $1.25kg$. Figure 2 shows that the expected profits decrease as the standard deviation of Y increases. The computational results agree with our intuition that expected profit $E(P_{T,III})$ for the two-stage screening is somewhat higher than that of the single-stage screening. $E(P_{S,II})$ is larger than $E(P_{S,I})$ if the standard deviation of Y is small, but $E(P_{S,II})$ is smaller than $E(P_{S,I})$ if the standard deviation of Y is large. Figure 3 indicates that μ_y^* tends to increase as the standard deviation of Y increases.

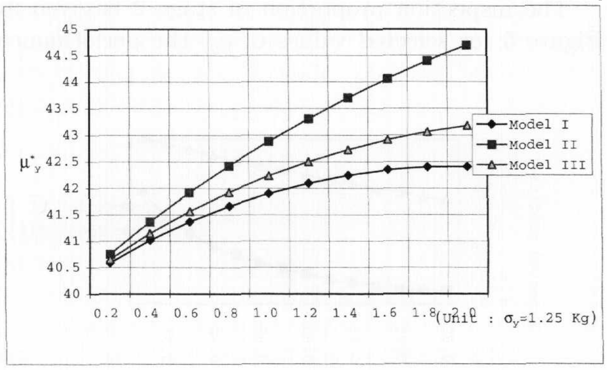

FIGURE 3. Optimal Process Mean as a Function of σ_y.

TABLE 1. Effects of ρ and σ on the optimal Process Mean, Screening Limits, and Expected Cost.

Models		Model II			Model III			
ρ	σ	μ_y^*	$\mathcal{X}^*$	$E(P_{S,II})$	μ_y^*	ω_1^*	ω_2^*	$E(P_{T,III})$
0.65	0.085	43.237	7.163	0.275	42.328	7.383	6.854	0.308
0.675	0.083	43.221	7.171	0.275	42.324	7.374	6.880	0.309
0.70	0.080	43.202	7.178	0.276	42.320	7.366	6.904	0.311
0.725	0.077	43.180	7.184	0.277	42.315	7.357	6.927	0.312
0.75	0.074	43.153	7.189	0.278	42.308	7.349	6.950	0.314
0.775	0.071	43.122	7.194	0.279	42.298	7.340	6.971	0.315
0.80	0.067	43.086	7.198	0.281	42.288	7.331	6.991	0.317
0.825	0.063	43.044	7.201	0.283	42.275	7.321	7.011	0.319
0.85	0.059	42.993	7.204	0.285	42.263	7.311	7.031	0.320
0.875	0.054	42.934	7.206	0.288	42.248	7.300	7.050	0.322
0.90	0.049	42.864	7.206	0.291	42.230	7.288	7.069	0.324
0.925	0.043	42.777	7.206	0.296	42.210	7.274	7.089	0.326
0.95	0.035	42.664	7.204	0.302	42.185	7.258	7.109	0.328
0.975	0.025	42.513	7.199	0.310	42.153	7.236	7.133	0.330

Effect of ρ

The expected profit per item and the optimum process mean and the screening limits on X are given in Table 1 and Figure 4 for selected values of ρ from 0.650 to 0.975. Table 1 shows that $E(P_{S,II})$ and $E(P_{T,III})$ increase as ρ increases. As ρ increases, the values μ_y^* and δ_1^* tend to decrease and the values of $\mathcal{X}^*$ and δ_2^* tend to increase. The value $E(P_{T,III})$ is greater than $E(P_{S,II})$ and the differences between $E(P_{T,III})$ and $E(P_{S,II})$ tend to decrease as ρ increases; these computational results agree with our expectations.

Effect of c_y

The inspection proportion in stage 2 is given in Figure 5 for selected values of c_y, the performance inspection cost, from 0.02 to 0.07. The computational results agree with our intuition that the inspection proportion in stage 2 tends to decrease as c_y increases.

Concluding Remarks

We have considered economic selections of the optimum mean value of the quality characteristic and the screening limits for a correlated variable under single and two-stage screenings. Models I and II for single-stage screening and Model III for two-stage screening are constructed under the assumption that the quality characteristic and the correlated variable

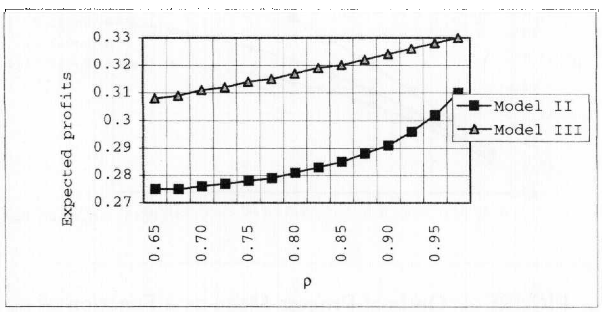

FIGURE 4. Expected Profit as a Function of ρ.

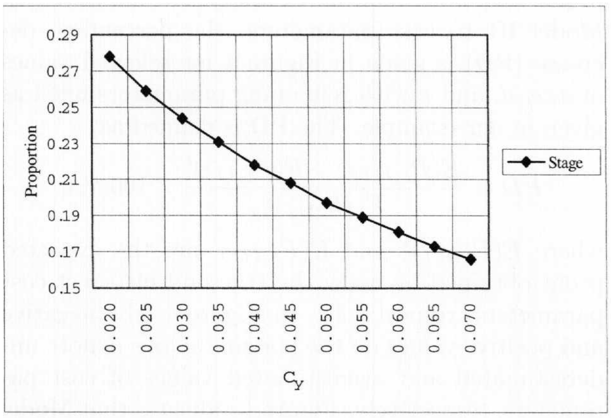

FIGURE 5. Proportion of Inspection as a Function of c_y.

are jointly normally distributed. The optimum process mean and screening limits are jointly obtained by maximizing the expected profit function, which includes the selling and discounted prices, in addition to the production, inspection, and penalty costs. For Model I, a closed form solution is obtained. For Models II and III the solutions are shown to be unique under the reasonable assumption that the proportion of defective items is less than 50%. However, closed form expressions for the optimum values are not obtained, and a numerical search algorithm such as Gauss-Siedel's iterative method is used. Numerical results show that the expected profit is more sensitive to selling and discounted prices than other cost parameters, and they show that the models are robust to changes in the cost parameters except for underestimated and overestimated values of selling and discounted prices. Expected profit decreases as the standard deviation of Y increases, and the process mean and screening limits on the correlated variable tend to increase as the standard deviation of Y increases. The expected profit for the two-stage screening procedure is somewhat greater than that of the single-stage screening procedures. As anticipated, the expected profits for Models II and III increase as ρ increases.

Appendix A: Derivation of Equation (8)

The expected profit per item is given by

$$E(P_{T,III}) = \int_{\omega_1}^{\infty}\int_{L}^{\infty}(a-b-cy-c_x)g(x,y)dydx + \int_{\omega_1}^{\infty}\int_{-\infty}^{L}(a-b-cy-c_x-d)g(x,y)dydx + \int_{\omega_2}^{\omega_1}\int_{L}^{\infty}(a-b-cy-c_x-c_y)g(x,y)dydx + \int_{\omega_2}^{\omega_1}\int_{-\infty}^{L}(r-b-cy-c_x-c_y)g(x,y)dydx + \int_{-\infty}^{\omega_2}\int_{-\infty}^{\infty}(r-b-cy-c_x)g(x,y)dydx. \quad \text{(A1)}$$

Using Equations (4) and (5) and the relationships

$$\int_{\omega_2}^{\omega_1}\int_{-\infty}^{L} g(x,y)dydx = \Psi(-\delta_2,\eta;-\rho) - \Psi(-\delta_1,\eta;-\rho),$$

$$\int_{\omega_2}^{\omega_1}\int_{L}^{\infty} g(x,y)dydx = \Psi(-\delta_2,-\eta;\rho) - \Psi(-\delta_1,-\eta;\rho),$$

Equation (A.1) can be rewritten as Equation (8).

Appendix B: Model III Derivations

The first derivatives of $E(P_{T,III})$ with respect to η and δ_i, $i=1,2$ are

$$\frac{\partial E(P_T)}{\partial \eta} = (a-r)\left[\Phi\left(\frac{\delta_2-\eta\rho}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\delta_1-\eta\rho}{\sqrt{1-\rho^2}}\right)\right]\phi(-\eta) - d\Phi\left(\frac{\delta_2-\eta\rho}{\sqrt{1-\rho^2}}\right)\phi(-\eta) - c\sigma_y - d\Phi\left(\frac{-\delta_1+\eta\rho}{\sqrt{1-\rho^2}}\right)\phi(-\eta) + c\sigma_y, \quad \text{(B1)}$$

$$\frac{\partial E(P_{T,III})}{\partial \delta_1} = (d+r-a)\Phi\left(\frac{\eta-\delta_1\rho}{\sqrt{1-\rho^2}}\right)\phi(\delta_1) - c_y\phi(\delta_1), \quad \text{(B2)}$$

and

$$\frac{\partial E(P_{T,III})}{\partial \delta_2} = (r-a)\Phi\left(\frac{-\eta+\delta_2\rho}{\sqrt{1-\rho^2}}\right)\phi(\delta_2) + c_y\phi(\delta_2). \quad \text{(B3)}$$

Setting Equations (B1)-(B3) to zero, we obtain Equations (9)-(11). The second partial derivatives of $E(P_{T,III})$ with respect to η and δ_i, $i=1,2$ at $(\eta^*,\delta_1^*,\delta_2^*)$ are

$$\frac{\partial^2 E(P_{T,III})}{\partial \eta^2} = (r-a)\rho\phi\left(\frac{\delta_2^*-\eta^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(-\eta^*)}{\sqrt{1-\rho^2}} - (d+r-a)\rho\phi\left(\frac{-\delta_1^*+\eta^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(\eta^*)}{\sqrt{1-\rho^2}} + c\sigma_y\eta^*,$$

$$\frac{\partial^2 E(P_{T,III})}{\partial \delta_1^2} = (a-d-r)\rho\phi\left(\frac{\eta^*-\delta_1^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(\delta_1^*)}{\sqrt{1-\rho^2}}$$

$$\frac{\partial^2 E(P_{T,III})}{\partial \delta_2^2} = (r-a)\rho\phi\left(\frac{-\eta^*+\delta_2^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(\delta_2^*)}{\sqrt{1-\rho^2}}$$

$$\frac{\partial^2 E(P_{T,III})}{\partial\eta\partial\delta_1} = \frac{\partial^2 E(P_{T,III})}{\partial\delta_1\partial\eta} = (d+r-a)\phi\left(\frac{\eta^* - \delta_1^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(\delta_1^*)}{\sqrt{1-\rho^2}}$$

$$\frac{\partial^2 E(P_{T,III})}{\partial\eta\partial\delta_2} = \frac{\partial^2 E(P_{T,III})}{\partial\delta_2\partial\eta} = (r-a)\phi\left(\frac{-\eta^* + \delta_2^*\rho}{\sqrt{1-\rho^2}}\right)\frac{\phi(\delta_2^*)}{\sqrt{1-\rho^2}}$$

$$\frac{\partial^2 E(P_{T,III})}{\partial\delta_1\partial\delta_2} = \frac{\partial^2 E(P_{T,III})}{\partial\delta_2\partial\delta_1} = 0.$$

Let H^2 and H^3 denote the matrix of second order partial derivatives evaluated at (η^*, δ_1^*) and $(\eta^*, \delta_1^*, \delta_2^*)$, respectively. Since $d \geq a > r$, it is clear that $\partial^2 E(P_{T,III})/\partial\eta^2 < 0$, $\partial^2 E(P_{T,III})/\partial\delta_1^2 < 0$, $\partial^2 E(P_{T,III})/\partial\delta_1^2 < 0$, and $\det(H^2) > 0$ for $\eta^* \leq 0$. It is difficult to show analytically that $\det(H^3) < 0$. However, extensive numerical studies over a wide range $[10 \leq a/(c\sigma_y) \leq 100,\ 5 \leq r/(c\sigma_y) \leq 50,\ 25 \leq d/(c\sigma_y) \leq 250,\ 0.6 \leq \rho \leq 0.99$, and $r/(c\sigma_y) < a/(c\sigma_y) < d/(c\sigma_y)]$ of parameter values $(r/(c\sigma_y), a/(c\sigma_y), d/(c\sigma_y), \rho)$ indicate that $\det(H^3) < 0$, suggesting that, over this range, the Hessian matrix is negative definite and that $E(P_{T,III})$ is a unimodal function of η and δ_i, $i = 1, 2$, for $\eta < 0$. Therefore, $(\eta^*, \delta_1^*, \delta_2^*)$ represents a maximum point of $E(P_{T,III})$ over a wide range of parameter values.

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